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Universality in multidimensional maps

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Abstract. Bifurcations of cubic nonlinear symplectic mappings in two and four dimensions are discussed. Orbits and types of bifurcations have been the subject of the preceding paper. Here series of bifurcations are studied by direct numerical calculation and by a renormalisation procedure. It is shown that for period-doubling bifurcations one finds the universal exponent of the quadratic area-preserving map. Other exponents exist for higher multiplicities. The renormalisation transformation has a fixed line in parameter space with an end point. The latter implies that series of period-doubling bifurcations may break off.

1. Introduction

In the last few years much interest has been paid to bifurcations of nonlinear maps, especially dissipative ones in one and two dimensions and area-preserving ones in two dimensions. The main part of these studies was devoted to the universal behaviour of period-doubling bifurcations (Feigenbaum 1978, 1979, Collet and Eckmann 1980, Bountis 1981, Greene *et al* 1981, Helleman 1980, Bak and Hoegh Jensen 1982, Janssen and Tjon 1982a (to be referred to as II)), but bifurcations with higher multiplicity have also been studied (Derrida and Pomeau 1980, II). In the preceding paper (Janssen and Tjon 1983, to be referred to as I) we have studied two- and four-dimensional symplectic mappings, which have their origin in a model in crystal physics (Janssen and Tjon 1981, 1982b).

The two- and four-dimensional mappings are given by

$$v = (x_n, x_{n-1}) \to Sv = ((\alpha-2)x_n + x_n^3 - x_{n-1}, x_n) \tag{1.1}$$

and by

$$v = (x_{n+1}, x_n, x_{n-1}, x_{n-2}) \to Sv = (x_{n+2}, x_{n+1}, x_n, x_{n-1}) \tag{1.2}$$

where

$$x_{n+2} = (2-\alpha-3\delta)(x_n/\delta) - (x_n^3/\delta) - [2-(1/\delta)](x_{n+1}+x_{n-1}) - x_{n-2} + c \tag{1.3}$$

for some constant c which will be taken to be zero in the following.

The orbits and bifurcations of these mappings have been discussed in I. It has been shown in a number of examples that one can distinguish several types of bifurcations. There are three types in which new fixed points originate from a parent one, whereas the character of the latter changes.

Type (a). A pair of eigenvalues of the linearised mapping moves over the unit circle in the complex plane, reaches $\lambda = +1$ and continues along the real axis; a new fixed point is created of the same order.

Type (b). A pair of eigenvalues reaches $\lambda = -1$ and continues along the real axis; a new fixed point is created with double the order.

Type (c). Two pairs of eigenvalues collide and leave the unit circle at $\lambda = \exp(\pm 2\pi i s/N)$ with s and N integers; new fixed points are created with the N-fold period.

There are two types where the eigenvalues reach $\lambda = \exp(\pm 2\pi i s/N)$ but stay on the unit circle.

Type (d). Two or more new fixed points originate from the parent one with N times the original period; this is called Birkhoff bifurcation.

Type (e). In the neighbourhood of a fixed point $2N$ new fixed points are created; N of them move away and bifurcate again, the other N move towards the parent fixed point, collide for the value of the parameter for which $\lambda = \exp(\pm 2\pi i s/N)$ and bounce back.

In I only the types of bifurcations were discussed. Often these occur in infinite series which have a geometric character. This is also the case for the four-dimensional mapping. In this paper we study in some detail the period-doubling bifurcations and evidence is given that the characteristic exponents are the same for both mappings. In § 2 we discuss infinite series of bifurcations, mainly period-doubling ones in four dimensions. The limiting behaviour can be studied using a renormalisation type of approach. In this way the universality of the exponents with respect to the dimensionality of the map can be understood. This is done in § 3.

2. Series of bifurcations

In I we discussed several types of bifurcations. Very often these bifurcations follow each other in series: if new cycles are created at a bifurcation these may bifurcate in turn and so on. In this respect there is a large difference from the one-dimensional map, where nearly every point in the unit interval is attracted to one cycle. In the two- and four-dimensional maps more and more cycles appear of different order for decreasing α. The domain of each cycle, i.e. the region in R^2 or R^4 where the motion is approximately described by the linearisation, may vary from cycle to cycle, but for sufficiently low values of α there is an infinite number of cycles in the plane with wildly varying values of period and winding number.

Feigenbaum (1978, 1979) has observed for the one-dimensional map that the period-doubling bifurcations occur very often in series and that the values of the parameter for which the bifurcation takes place form a geometric series. Moreover, he found that the rate of this series is a universal constant independent of the specific map. An analogous behaviour has been found for the area-preserving map of the plane. Also here there is a universal exponent for period-doubling bifurcations.

In II we have discussed two series of period-doubling bifurcations for $\delta = 0$: one starts from the non-trivial FP $N = 1$ which exists for $\alpha < -2$, the other starts via a bifurcation of type (e) from the solution with period $N = 2$ at $\alpha = -1$. If α_n is the value of α for which the eigenvalue of the cycle with period $N = 2^n$ becomes $\lambda = -1$

and a bifurcation of type (*b*) takes place, one can define (following Feigenbaum)

$$\eta_n = \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+2} - \alpha_{n+1}} \tag{2.1}$$

and one finds that η_n tends to $\eta_\infty = 8.721\ldots$ for $n \to \infty$. For $\delta \neq 0$ one finds series of period-doubling bifurcations that are obtained smoothly from those at $\delta = 0$. A number of series of bifurcations is given in table 1 which shows that the rate of the geometric series is in agreement with the value for the two-dimensional map, which is a universal constant as shown by Eckmann (private communication). (For the first two series in table 1 only periods up to $N = 32$ are given. The solutions of longer period are difficult to find numerically, because the third eigenvalue goes to infinity.)

Table 1. Series of period-doubling bifurcations. α_m is the value for which two eigenvalues of the $N = 2^m$ cycle become -1; η_m is the ratio (6.1).

	$\delta = 0.45$		$\delta = 1.0$		$\delta = 3.059$	
N	α_m	η_m	α_m	η_m	α_m	η_m
2	−0.65	—				
2	−1.215 08	—	−7.196 1	—	−4.059	—
4	−1.317 53	5.52	−7.243 108	—	−4.312 783	—
8	−1.329 484	8.570	−7.252 143 4	5.20	−4.463 779 7	1.681
16	−1.330 856 9	8.707	−7.253 223 85	8.36	−4.480 507 09	9.027
32	−1.331 014 3	8.721	−7.253 348 147	8.69	−4.482 388 61	8.890
64					−4.482 607 742	8.586
128					−4.482 632 643 6	8.800

In the limit of $\delta \to 0$ the orbits in R^4 are related to those for $\delta = 0$. Another value of δ for which the mapping is essentially two-dimensional is $\delta = 0.5$. For this value it is easily seen from equation (1.3) that the chain $\{x_n\}$ consists of two subchains: those of the odd and those of the even positions. When there is a bifurcation series for the mapping in R^2 with bifurcation points α'_m, then there is a corresponding bifurcation series for $\delta = 0.5$ with $\alpha_m = \frac{1}{2}(\alpha'_m - 1)$. Such a bifurcation series exists already if one of the two subchains has one. This means that one such series can give rise to several for $\delta = 0.5$. The value $\alpha_\infty = -1.63\ldots$ found for $\delta = 0.5$, for example, corresponds to the value $\alpha_\infty = -2.27\ldots$ at $\delta = 0$ (cf II). Actually there are several solutions with the same subchain. Therefore the point $\alpha = -1.63\ldots$, $\delta = 0.5$ is a point where a number of lines $\alpha = \alpha_\infty(\delta)$ cross. If $\delta \neq 0.5$ this degeneracy is lifted.

An essential difference between the two- and four-dimensional mappings is that for the latter a series of bifurcations may terminate because the eigenvalues of the bifurcated cycle do not reach the same position on the unit circle as those of the parent cycle at the bifurcation, but leave the unit circle earlier. This is the reason why the period-doubling sequence of table 1 does not continue indefinitely for $\delta > 4.5$.

Series of bifurcations also exist from just one cycle, instead of from the successively bifurcated cycles. Consider as an example the cycle of period $N = 4$ with $\{x_n = a, a, -a, -a\}$ for $\delta = 0$ $(a^2 = 2 - \alpha)$. For $\alpha = 1$ this cycle has an eigenvalue $\lambda = +1$; for $\alpha = \alpha_m = 1 + \varepsilon_m$ a series of bifurcations takes place to cycles with periods $N' = 4 \times 2^m$.

For m tending to ∞, one can approximate α_m by

$$\alpha_m \approx 1+2\pi^2/4^m. \tag{2.2}$$

Hence the rate of this geometric series is four. Another series with the same rate and periods $N=4^m$ is given in table 2. The situation here is a succession of bifurcations of type (d). If for a given value of α and δ there is an eigenvalue $\exp(2\pi is/N)$ where s is the winding number, a new fixed point of order $4N$ may bifurcate. The orbit resembles that of one with period equal to the nearest integer to N/s. For the second example in table 2 this is the series $N/s=4, \frac{16}{5}, \frac{64}{21}, \frac{256}{85}, \frac{1024}{341}$, etc, which tends to 3. The corresponding values of α_m tend to the value of α for which the bifurcation with $\lambda=\exp(\frac{2}{3}\pi i)$ takes place, i.e. $\alpha=1.5$. In this case it can easily be shown analytically that these values of α_m form a geometric series with rate 4 (see appendix).

Table 2. Two series of bifurcations for $\delta=0$. Both series correspond to antisymmetric solutions and depart from the $N=4$ cycle $x_n=(a, a, -a, -a)$. α_m is the value for which the period $4p^m (p=2, 4)$ has an eigenvalue $\lambda=-1$.

	First series			Second series		
m	N	α_m	η_m	N	α_m	η_m
0	4	1.292 9	—	4	1.617 32	—
1	8	1.076 123 9	—	16	1.528 604	—
2	16	1.019 215 0	3.81	64	1.507 102	4.126 1
3	32	1.004 815 4	3.95	256	1.501 772 3	4.034 1
4	64	1.001 204 55	3.988	1024	1.500 442 88	4.009 0
5	128	1.000 301 19	3.998			
∞		1.0	4		1.5	4

If one considers one series of period-doubling bifurcations for $\delta \neq 0$ the accumulation point for the values α_m can be denoted by $\alpha_\infty(\delta)$. Because the solutions depend smoothly on δ one obtains a line in the α, δ plane which we call the critical line. The values of α for which the cycle of period 2^m bifurcates into a cycle of period 2^{m+1} are also situated on smooth lines $\alpha_m(\delta)$ which approach the critical line geometrically. The interesting fact is that the rate of the geometric series does not depend on δ. For $\delta=\delta_0+\varepsilon$ one has the relation

$$\alpha_n(\delta)=\alpha_\infty(\delta)+A(\delta)\eta^{-n}=\alpha_n(\delta_0)+\varepsilon(c_1+c_2\eta^{-n}) \tag{2.3}$$

where c_1 and c_2 are constants. The qualitative behaviour of such a critical line is sketched in figure 1. For any direction intersecting this critical line the bifurcations form a geometric series with the same exponent. However, depending on the conditions, the convergence may be very different. Two examples are given in table 3. One series proceeds along a straight line, the other connects the bifurcation points for which the cycle of period $N=2^m$ has two eigenvalues $\lambda=+1$ and two eigenvalues $\lambda=-1$ and a cycle of period $N'=2^{m+1}$ is created. These values α_m, δ_m should form also a geometric series with a rate 8.7. The convergence of the second series, however, is much slower than that of the first.

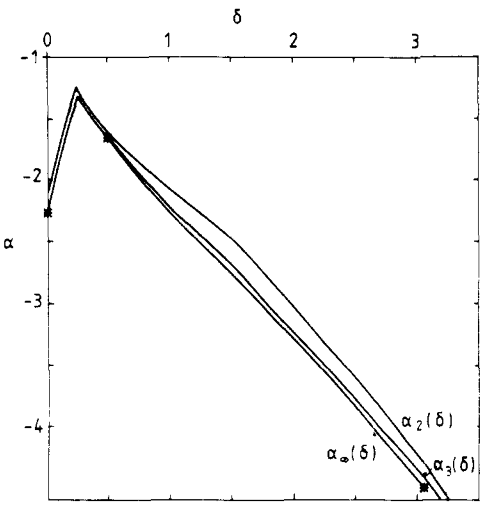

Figure 1. The first few curves $\alpha_p(\delta)$ on which one of the eigenvalues of DS^N for $N=2^p$ has the value -1. The curves approach the critical curve $\alpha_\infty(\delta)$ geometrically. The asterisks indicate the fixed points of the renormalisation transformation (§ 3). The FP near $\alpha=-4.5$, $\delta=3$ is an end point of the critical line. The sharp bend is the cross-over at $\delta=0.25$ from series starting with an $N=2$ configuration $\{a, b\}$ to one starting from $(a, -a)$.

Table 3. Two series of period-doubling bifurcations converging to the same α, δ values: (a) along $\alpha=\alpha_c-2.7(\delta-\delta_c)$; ($b$) the points where $\lambda_1=\lambda_2=-1$, $\lambda_3=\lambda_4=+1$.

(a) $\alpha=\alpha_c-2.7\,(\delta-\delta_c)$			
N	α_m	δ_m	η_m
4	−4.151 034	2.937 797	—
8	−4.448 428	3.047 943	—
16	−4.482 893 8	3.060 708 4	8.629
32	−4.486 797 68	3.062 154 29	8.829
64	−4.487 252 766	3.062 322 848	8.578
128	−4.487 304 474 6	3.062 342 000	8.801
256	−4.487 310 435 79	3.062 344 207 6	8.674
512	−4.487 311 117 185	3.062 344 459 96	8.749

(b) $\lambda_1=-1, \lambda_3=+1$			
N	α_m	δ_m	η_m
4	−3.657 5	2.552 4	—
8	−4.429 608 9	3.034 371	—
16	−4.478 858 86	3.057 820	15.67
32	−4.486 464 53	3.061 916	6.47
64	−4.487 207 818	3.062 290 7	10.23
128	−4.487 301 482 8	3.062 339 86	7.94
256	−4.487 311 582 86	3.062 345 029	9.27

3. Renormalisation calculation

To study the exponents of the geometric series one can also use a generalisation of the renormalisation group approach first suggested by Derrida and Pomeau (1980). Consider a series of period-doubling bifurcations along a line in the α, δ plane. If α_m, δ_m are the values of the parameters for which a cycle of period $N = 2^m$ bifurcates and gives a cycle of period $N' = 2^{m+1}$, the values α_m, δ_m form a geometric series which has the point $\alpha_\infty, \delta_\infty$ as a limit. The eigenvalues λ_i depend smoothly on the parameters, if one stays with the same solution, or its bifurcates at the bifurcation points. We denote the eigenvalues of DS^{2^m} for one specific series of solutions by $\lambda_i(m, \alpha, \delta)$. Because the bifurcations are of type (b) one has

$$\lambda_1(m, \alpha_m, \delta_m) = -1. \tag{3.1}$$

In the neighbourhood of $\alpha_\infty, \delta_\infty$ the other eigenvalues λ_3 and λ_4 do not change rapidly. So one has only the effect of the doubling of the period:

$$\lambda_3(m, \alpha_m, \delta_m)^2 \approx \lambda_3(m+1, \alpha_{m+1}, \delta_{m+1}). \tag{3.2}$$

This means that we are looking for bifurcations at points in parameter space where the character of all eigenvalues is the same. Starting from an arbitrary value of α, δ in the neighbourhood of $\alpha_\infty, \delta_\infty$ one can determine a series $\alpha, \delta \to \alpha^{(1)}, \delta^{(1)} \to \alpha^{(2)}, \delta^{(2)} \dots$ by

$$\begin{aligned} &\lambda_1(m, \alpha^{(m)}, \delta^{(m)}) = \lambda_1(m+1, \alpha^{(m+1)}, \delta^{(m+1)}) \\ &\lambda_3(m, \alpha^{(m)}, \delta^{(m)})^2 = \lambda_3(m+1, \alpha^{(m+1)}, \delta^{(m+1)}). \end{aligned} \tag{3.3}$$

This series will converge towards $\alpha_\infty, \delta_\infty$. One can approximate this mapping by a mapping from α, δ to α', δ' defined by

$$\begin{aligned} &\lambda_1(m, \alpha, \delta) = \lambda_1(m+1, \alpha', \delta') \\ &\lambda_3(m, \alpha, \delta)^2 = \lambda_3(m+1, \alpha', \delta') \end{aligned} \tag{3.4}$$

for a fixed value of m. We are considering here only one family of solutions. For a different family, even with the same periods and winding numbers, the transformation in parameter space may be quite different. This mapping is a kind of renormalisation transformation. The values $\alpha_\infty, \delta_\infty$ can be approximated by the fixed points (FP) of this transformation.

Numerically one can search for FP of the transformation (3.4). We have performed the calculation for $m = 2$ and $m = 3$ (see table 4). One FP for $m = 3$ (other values are given in table 4) in a series starting from the antisymmetric solution with period 2,

Table 4. The fixed points of the transformation (3.12) for different values of m in one family of solutions and the eigenvalues of the linearised transformation around the fixed points.

N	N'	m	α	δ	η_1^{-1}	η_2^{-1}
8	16	3	−4.474 1	3.052 7	7.98	4.19
16	32	4	−4.488 230 1	3.063 01	8.98	3.91
4	8	2	−1.637 255 2	0.5	8.420	0.928
8	16	3	−1.637 257 5	0.5	8.727	1.000

where its eigenvalue is -1, is

$$\alpha_c \approx -1.637\,25 \qquad \delta_c = 0.5. \tag{3.5}$$

In this family the eigenvalue λ_3 goes to infinity if m increases. Linearisation of the renormalisation transformation gives a linear mapping from $\alpha_c+\Delta\alpha$, $\delta_c+\Delta\delta$ to $\alpha_c+\Delta\alpha'$, $\delta_c+\Delta\delta'$ with

$$\begin{pmatrix}\Delta\alpha' \\ \Delta\delta'\end{pmatrix} = R\begin{pmatrix}\Delta\alpha \\ \Delta\delta\end{pmatrix} \tag{3.6}$$

where R can be expressed in terms of the derivatives of the eigenvalues λ_1 and λ_3 for $N = 2^m$ and λ_1' and λ_3' for $N' = 2^{m+1}$:

$$R = R_1 R_2^{-1} \tag{3.7}$$

with

$$R_1 = \begin{pmatrix}\partial\lambda_1/\partial\alpha & \partial\lambda_3/\partial\alpha \\ \partial\lambda_1/\partial\delta & \partial\lambda_3/\partial\delta\end{pmatrix}$$

$$R_2 = \begin{pmatrix}\partial\lambda_1'/\partial\alpha & \partial\sqrt{\lambda_3'}/\partial\alpha \\ \partial\lambda_1'/\partial\delta & \partial\sqrt{\lambda_3'}/\partial\delta\end{pmatrix}.$$

The eigenvalues of R for the FP given in (3.5) are $\eta_1^{-1} = 1/8.727\ldots$ and $\eta_2 = 1.0000$. This means that the line tangent to the eigenvector of $\eta = 1$ is a fixed line. The reason is that the points of the line $\alpha = \alpha_\infty(\delta)$ satisfy the equation

$$\lambda_1(m, \alpha, \delta) = \lambda_1(m+1, \alpha, \delta) \tag{3.8}$$

in the limit of $m \to \infty$. So for large m the renormalisation transformation maps every point of the critical line to a point on this line. Since in this case the second condition in equation (3.4) is automatically satisfied the critical line is left invariant point-wise. This is equivalent to the statement that $\partial\lambda_3/\partial\alpha = \partial\sqrt{\lambda_3'}/\partial\alpha$ and similarly for δ, because then R has an eigenvalue $+1$. The eigenvalue η_1^{-1} corresponds to the exponent already known for the two-dimensional mapping.

Two other FP of the renormalisation transformation are

$$\begin{aligned} \alpha_c &= -1.084\,593\ldots \qquad & \delta_c &= 0 \\ \alpha_c &= -2.274\,515\ldots \qquad & \delta_c &= 0. \end{aligned} \tag{3.9}$$

These points correspond to the FP of two renormalisation transformations for two different families of solutions, starting from $\alpha = -1$ and $\alpha = -2$, respectively, along the α axis for the two-dimensional mapping

$$\lambda(m, \alpha, 0) = \lambda(m+1, \alpha', 0'). \tag{3.10}$$

This can be seen from the fact that if δ tends to zero, one eigenvalue of a FP goes to infinity as δ^{-m-1}, whereas another eigenvalue tends to the eigenvalue of the two-dimensional mapping. In II we have shown that the values (3.9) are the limiting values α_∞ for a series of period-doubling bifurcations.

Another series of period-doubling bifurcations is obtained if one considers the values of α and δ for which

$$\lambda_1(m, \alpha_m, \delta_m) = -1 \qquad \lambda_3(m, \alpha_m, \delta_m) = +1. \tag{3.11}$$

This is a series for which $\lambda_3 = \lambda_3'$ at the bifurcation points. One can approximate the values of α_∞ and δ_∞ by the FP of the renormalisation transformation (3.4). One finds a FP for $\alpha = -4.449\ldots$, $\delta = 3.035\ldots$. The eigenvalues of the linearised renormalisation transformation are $9.0\ldots$ and $1.0\ldots$. The latter implies that the critical line is also a fixed line here. This critical line runs from this FP via the FP (3.5) to one at $\delta = 0.25$. The fact that numerically one finds FP and not a fixed line is due to the approximation made. The existence of a (point-wise) fixed line explains why the other exponent is universal: there is only one relevant parameter.

A numerically more practical way to find the FP more precisely is one based on the observation that it is also a FP of another transformation:

$$\lambda_k(m, \alpha, \delta) = \lambda_k(m+1, \alpha', \delta') \qquad k = 1, \ldots, 4. \tag{3.12}$$

We have determined the FP for $m = 3$ and $m = 4$ numerically (table 4). For $m = 4$ a FP is

$$\alpha_c = -4.488\,230\,1\ldots \qquad \delta_c = 3.063\,01\ldots. \tag{3.13}$$

The eigenvalue λ_3 now tends to one as m tends to infinity. Linearising the renormalisation transformation around α_c, δ_c gives the eigenvalues η_k^{-1} and eigenvectors of R:

$$\begin{aligned} &\eta_1 = 8.89\ldots \qquad \text{eigenvector: } (-2.68, 1) \\ &\eta_2 = 3.91\ldots \qquad \text{eigenvector: } (-1.39, 1). \end{aligned} \tag{3.14}$$

The first exponent is consistent with the universal exponent $8.72\ldots$, although the error is still rather large. The direct calculation varying α, δ along a straight line gives better agreement (table 3). The critical line $\alpha = \alpha_\infty(\delta)$ is again a solution of $\lambda_1(m, \alpha, \delta) = -1$ in the limit of $m \to \infty$. Hence this line is an invariant line, along which no period-doubling bifurcations take place. At the FP the critical line is tangent to the second eigenvector. The motion of the points on the line is governed by a different exponent which is approximately equal to four.

The above results suggest that the period-doubling bifurcations in R^4 can also be described by the universal exponent of $8.721\ldots$ for the rate of the geometric series. This is not so surprising in the neighbourhood of $\delta = 0$ and $\delta = 0.5$ where the mapping is nearly two-dimensional. For other values of δ, although each FP of the renormalisation group transformation has more than one exponent, only one is relevant for the bifurcation; the other one describes the motion along the critical line.

4. Concluding remarks

The cubic mapping in two and four dimensions shows a large number of geometric series of bifurcations, where the period of a solution becomes a multiple of the original one. In II we considered some of the series of bifurcations with $N = ap^m$ ($m = 1, 2, \ldots$) for $\delta = 0$. It was shown that (i) these bifurcations give also a geometric series for the values of α at which the bifurcations take place, (ii) the rate of the geometric series is, in general, different from that for the period-doubling series, (iii) the rate depends not only on the multiplicity p of the bifurcation but also on the specific series. Since the solutions depend smoothly on α and δ one may expect that the values for higher-order bifurcations found for $\delta = 0$ should also apply to the case $\delta \neq 0$ but we did not investigate this point thoroughly.

The universality of the exponent for period-doubling bifurcations can be understood for the present mapping from the existence of a critical line in the parameter space which is a fixed line for a renormalisation transformation where the fixed points are the accumulation points of Feigenbaum sequences. Since the critical line is point-wise fixed only one exponent is relevant. As a special feature in our model, this critical line has an end point: for a large enough value of the parameter δ, series of period-doubling bifurcations break off. This is because the nonlinear terms are less important for large δ.

Appendix

Here we consider series of bifurcations at $\delta = 0$ from the $N = 4$ cycle $\{x_n = a, a, -a, -a\}$ where $a^2 = 2-\alpha$. A bifurcation of type (d) may take place if the eigenvalue of DS^N is $\exp(2\pi i s/p)$. Then a cycle with period $N = 4p$ may branch off. Now we consider a series of rational numbers $s/p = s_m/p_m$. This gives a series of bifurcations from $N = 4$ to $N = 4p_m$. Suppose now that the series s_m/p_m converges to the rational number k/l in a geometric way:

$$s_m/p_m = k/l + A\theta^{-m}. \tag{A1}$$

The eigenvalues of DS^4 are determined by its trace:

$$T = 2\cos\phi = (4-2\alpha)^4 - 4(4-2\alpha)^2 + 2. \tag{A2}$$

If $\phi_m = 2\pi s_m p^{-m}$ then (A2) determines the values α_m for which the bifurcation takes place and $\phi = 2\pi k/l$ determines the accumulation point α_∞. For small values of $A\theta^{-m}$ one may write $\alpha_m = \alpha_\infty + \varepsilon_m$ and

$$2\cos\phi_\infty - 2\sin\phi_\infty A\theta^{-m} \approx T(\alpha_\infty) + \varepsilon_m[4(4-2\alpha_\infty)^3 - 8(4-2\alpha_\infty)]. \tag{A3}$$

If $\sin\phi_0 \neq 0$ this gives an expression for ε_m:

$$\alpha_m = \alpha_\infty + \frac{2A\sin\phi_\infty}{8(4-2\alpha_\infty) - 4(4-2\alpha_\infty)^3}\theta^{-m} \tag{A4}$$

which means that the values of α_m converge towards α_∞ with a rate θ. As an example consider the series $s_m p^{-m} = (4^m - 1)/(3\times 4^m) = \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}\ldots$ converging to $k/l = \frac{1}{3}$ with rate $\theta = 4$ and $A = -\frac{1}{3}$. Hence the bifurcation points converge towards $\alpha_\infty = 1.5$ with rate 4. The orbits look more and more like that for period 3×4.

If $\sin\phi_0 = 0$, one has to take into account higher-order terms in the expansion:

$$2\cos\phi_\infty - (A\theta^{-m})^2 \approx T(\alpha_\infty) + \varepsilon_m[4(4-2\alpha_\infty)^3 - 8(4-2\alpha_\infty)]. \tag{A5}$$

Hence α_m converges in this case to α_∞ with rate θ^2. An example of this situation is $s_m p^{-m} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ converging to zero. Then $\alpha_\infty = 2$ or 1 and α_m converges to one of these values with rate $2^2 = 4$.

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